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Relativistic space–times having corresponding geodesics†

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Abstract. Pairs of relativistic space–times are classified according to their Segrè characteristics. Suitable bases consisting of pseudo-orthonormal tetrads are constructed and the condition that the spaces should have corresponding geodesics is imposed. It is found that the [3, 1] and [(3, 1)] classes contain spaces with corresponding geodesics. The most general forms of the metrics in these classes are derived. Of these metrics, the vacuum ones are shown to be algebraically special, in the sense of the Petrov classification.

1. Introduction

Let V_n and V_n' be two Riemannian n-spaces with fundamental forms g_{ab} and h_{ab}. If the elementary divisors of g_{ab} and h_{ab} are all real and simple, as is the case when g_{ab} is positive definite, then there exist n mutually orthogonal non-null eigenvectors. However, when both spaces are indefinite, null eigenvectors may occur and there is the possibility that the elementary divisors are not simple. In such cases the eigenvectors do not span the spaces. Relativistic space–times are Riemannian 4-spaces and may be spanned by eigenvectors and generalized eigenvectors.

Wong (1945) has developed the theory of quasi-orthogonal ennuples, which had previously been introduced by Lense (1932), and applied it to the problem of finding pairs of V_3 with corresponding geodesics. Bases consisting of eigenvectors and generalized eigenvectors forming quasi-orthonormal tetrad systems (Sachs 1961, Goldberg and Kerr 1961, Newman 1961) are here found to be suitable frameworks for the consideration of the problem in four dimensions as well.

The correspondence between the geodesics of the relativistic spaces would mean physically that motions of free particles would be in correspondence. The equations of test particles in the one space would also be the equations of test particles in the second space.

Of special interest are empty relativistic space–times having corresponding geodesics. Two spaces have corresponding geodesics if, and only if, their projective curvature tensors are identical (Eisenhart 1926). In empty space, since the projective curvature tensor and the conformal tensor are identical, the Petrov classification (Jordan *et al.* 1960) here gives a classification of spaces with corresponding geodesics.

2. Quasi-orthogonal tetrad

Let $\underset{(\alpha)}{v}{}^a$, $\alpha = 1 \ldots 4$‡ be a basis which forms a quasi-orthogonal tetrad in the space g_{ab}. The basis constructed in the [3, 1] and [(3, 1)] classes consists of two null vectors and two unit space-like vectors. If the null vectors are $\underset{(1)}{v}{}^a$ and $\underset{(2)}{v}{}^a$, the space-like vectors $\underset{(3)}{v}{}^a$ and $\underset{(4)}{v}{}^a$, then they satisfy the following quasi-orthogonal conditions:

$$\underset{(1)}{v}{}^a \underset{(2)}{v}{}_a = 1, \quad \underset{(1)}{v}{}^a \underset{(3)}{v}{}_a = 0, \quad \underset{(1)}{v}{}^a \underset{(4)}{v}{}_a = 0, \quad \underset{(1)}{v}{}^a \underset{(1)}{v}{}_a = 0, \quad \underset{(2)}{v}{}^a \underset{(2)}{v}{}_a = 0$$

$$\underset{(2)}{v}{}^a \underset{(3)}{v}{}_a = 0, \quad \underset{(2)}{v}{}^a \underset{(4)}{v}{}_a = 0, \quad \underset{(3)}{v}{}^a \underset{(4)}{v}{}_a = 0, \quad \underset{(3)}{v}{}^a \underset{(3)}{v}{}_a = 1, \quad \underset{(4)}{v}{}^a \underset{(4)}{v}{}_a = 1.$$

The signatures of the spaces are $+2$.

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‡ Latin indices denote tensor components and Greek indices tetrad components.

Let us define invariants $\underset{\alpha\beta}{g}$, $\underset{\alpha\beta}{h}$ and $\overset{\alpha\beta}{g}$:

$$\underset{\alpha\beta}{g} = g_{ab}\underset{(\alpha)}{v^a}\underset{(\beta)}{v^b}, \qquad \underset{\alpha\beta}{h} = h_{ab}\underset{(\alpha)}{v^a}\underset{(\beta)}{v^b}, \qquad \overset{\alpha\beta}{g}\underset{\beta\rho}{g} = \delta^{\alpha}_{\rho}.$$

Any tensor can be expressed in terms of the vectors $\underset{(\alpha)}{v^a}$ and some invariants. For example, a tensor of the third order A_{abc} can be expressed as

$$A_{abc} = \overset{\alpha\beta\rho}{A}\,\underset{(\alpha)}{v_a}\underset{(\beta)}{v_b}\underset{(\rho)}{v_c}.$$

In particular,

$$g_{ab} = \overset{\alpha\beta}{g}\,\underset{(\alpha)}{v_a}\underset{(\beta)}{v_b}$$

and

$$h_{ab} = \overset{\alpha\beta}{h}\,\underset{(\alpha)}{v_a}\underset{(\beta)}{v_b}$$

where h is defined to be $\overset{\alpha\rho}{g}\,\overset{\beta\lambda}{g}\,\underset{\rho\lambda}{h}$.

The matrix representation of h_{ab} relative to the quasi-orthogonal basis, in the space g_{ab}, is given by ${}_{\alpha}h^{\beta}$, where ${}_{\alpha}h^{\beta} = \overset{\rho\beta}{g}\,\underset{\alpha\rho}{h}$. The bases in each case will consist of eigenvectors and generalized eigenvectors of h_{ab} in the space g_{ab}. The matrix representations will thus be Jordan canonical forms, a unique representation for each Segrè case (Schouten 1954).

Coefficients of rotation (Eisenhart 1926) are a set of invariants $\underset{\alpha\beta\rho}{\gamma}$ defined by

$$\underset{\alpha\beta\rho}{\gamma} = \underset{(\alpha)}{v_{a|b}}\underset{(\beta)}{v^a}\underset{(\rho)}{v^b}.\dagger$$

They have the property that $\underset{(\alpha\beta)\rho}{\gamma} = 0.\ddagger$

The necessary and sufficient conditions for the congruence $\underset{(\alpha)}{v^a}$ to be hypersurface orthogonal are

$$\underset{(\alpha)}{v}_{[a}\underset{(\alpha)}{v}_{b|c]} = 0.$$

In terms of rotation coefficients these become, for a null congruence such as $\underset{(1)}{v^a}$,

$$\underset{131}{\gamma} = \underset{141}{\gamma} = 0$$

and, for a space-like congruence such as $\underset{(3)}{v^a}$,

$$\underset{3[\beta\rho]}{\gamma} = 0, \qquad \beta, \rho \neq 3.$$

Sufficient conditions for null congruences such as $\underset{(1)}{v^a}$ to be geodesic are

$$\underset{131}{\gamma} = \underset{141}{\gamma} = 0.$$

For a space-like congruence $\underset{(3)}{v^a}$ necessary and sufficient conditions for a geodesic are $\underset{\alpha 33}{\gamma} = 0$ for all α.

Expansion $\underset{(\alpha)}{\theta}$ is defined by $\underset{(\alpha)}{v^a}_{|a}$. Let $\underset{(\alpha)}{k}_{ab}$ be a projection operator, projecting into the infinitesimal 3-space orthogonal to the non-null vector $\underset{(\alpha)}{v^a}$ and $\underset{(\alpha)}{\dot{v}^a} = \underset{(\alpha)}{v^a}_{|b}\underset{(\alpha)}{v^b}$; then shear

† The vertical is used to denote covariant differentiation and a comma will be used for partial differentiation.

‡ Round brackets around two or more tensor or tetrad indices denote symmetry on the indices enclosed and square brackets will be used for skew symmetry.

$\underset{(\alpha)}{\sigma_{ab}}$ is defined by

$$\underset{(\alpha)}{\sigma_{ab}} = \underset{(\alpha)}{v_{(a|b)}} + \underset{(\alpha)}{\dot{v}_{(a}}\underset{(\alpha)}{v_{b)}} - \tfrac{1}{3}\,\theta\,\underset{(\alpha)(\alpha)}{k_{ab}} \quad \text{(Ehlers and Kundt 1962).}$$

In the case of $\underset{(\alpha)}{v^a}$ being null and geodetic the shear of the congruence is given by

$$\underset{(\alpha)}{|\sigma|^2} = \tfrac{1}{2}(\underset{(\alpha)}{v_{(a|b)}}\underset{(\alpha)}{v^{a|b}} - 2\theta^2).$$

3. [4] Segrè characteristic

The canonical matrix representation ${}_\alpha h^\beta$ of h_{ab} in the space g_{ab} is

$$\begin{pmatrix} A & 1 & 0 & 0 \\ 0 & A & 1 & 0 \\ 0 & 0 & A & 1 \\ 0 & 0 & 0 & A \end{pmatrix}.$$

Here the only eigenvalue is A, repeated three times.

Let the base vectors for this representation be z^a, y^a, t^a and x^a, defined by the following chain:

$$\begin{aligned} x^a &= (h^a{}_b - A\delta^a_b)y^b \\ t^a &= (h^a{}_b - A\delta^a_b)x^b \\ z^a &= (h^a{}_b - A\delta^a_b)t^b \\ (h^a{}_b - A\delta^a_b)z^b &= 0. \end{aligned}$$

z^a is an eigenvector; t^a, x^a and y^a are generalized eigenvectors of ranks 2, 3 and 4 respectively.

The relationships between these vectors will now be investigated:

$$z^a z_a = (h^a{}_b - A\delta^a_b)t^b z_a = 0$$

implying that z^a is null;

$$t^a z_a = (h^a{}_b - A\delta^a_b)x^b z_a = 0$$

implying that t^a and z^a are orthogonal;

$$\begin{aligned} t^a t_a &= (h^a{}_b - A\delta^a_b)x^b t_a = x^a z_a \\ &= (h^a{}_b - A\delta^a_b)y^b z_a = 0 \end{aligned}$$

implying that t^a is null. Since the spaces of interest are known not to admit real orthogonal null vectors this case can be excluded.

A similar approach was taken in each of the [2, 2] and [(2, 2)] Segrè classes. There it was found, in each case, that a pair of null, mutually orthogonal eigenvectors or generalized eigenvectors had to exist. The spaces of interest, being of signature +2, are known not to allow such vectors. Hence these classes contain no relativistic space–times having corresponding geodesics.

4. [3, 1] Segrè characteristic

The Jordan canonical matrix representation is

$$\begin{pmatrix} A & 1 & 0 & 0 \\ 0 & A & 1 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A+B \end{pmatrix}$$

A and $A+B$ being distinct eigenvalues. The base vectors form the chains

$$\left.\begin{aligned} x^a &= (h^a{}_b - A\delta^a_b)y^b \\ z^a &= (h^a{}_b - A\delta^a_b)x^b \\ (h^a{}_b - A\delta^a_b)z^b &= 0 \\ \{h^a{}_b - (A+B)\delta^a_b\}t^b &= 0 \end{aligned}\right\}. \tag{4.1}$$

z^a and t^a are eigenvectors, x^a is a generalized eigenvector of rank 2 and y^a is a generalized eigenvector of rank 3.

It will now be shown that a unique quasi-orthogonal ennuple of the type discussed in § 2 can be constructed satisfying these eigenvector conditions:

$$z^a z_a = (h^a{}_b - A\delta^a_b)x^b z_a = 0$$

implying that z^a is null;

$$z^a x_a = z^a(h^b{}_a - A\delta^b_a)y_b = 0$$

implying that x^a and z^a are orthogonal. x^a must therefore be space-like. Let us normalize x^a: $x^a x_a = 1$;

$$y^a z_a = y^a(h^b{}_a - A\delta^b_a)x_b = x^b x_b = 1.$$

We shall contract the third equation of the set (4.1) with t_a and the fourth with z_a. Subtraction, if we take into account the fact that $B \neq 0$, gives t^a as being orthogonal to z^a. t^a must therefore be space-like and can be normalized: $t^a t_a = 1$. Similarly the second and fourth equations of (4.1) lead to $x^a t_a = 0$, and the first and fourth to $y^a t_a = 0$.

y^a, being a generalized eigenvector of rank 3, may be used to construct the following general eigenvector of rank 3:

$$\bar{y}^a = \rho y^a + \mu x^a + \gamma z^a$$

where ρ, μ and γ are scalars. The remaining vectors in the chain $\bar{x}^a$ and $\bar{z}^a$ would then be defined by

$$\bar{x}^a = \rho x^a + \mu z^a, \qquad \bar{z}^a = \rho z^a.$$

Since $\bar{y}^a$ is a generalized eigenvector of rank 3, $\bar{x}^a$ a generalized eigenvector of rank 2 and $\bar{z}^a$ an eigenvector, all the identities previously found apart from the normalized results will be satisfied. To satisfy the condition $\bar{x}^a \bar{x}_a = 1$, ρ has to be unity. t^a, as defined in (4.1), can be chosen within a scalar multiple. However, the condition $t^a t_a = 1$ selects a unique scalar multiple. The freedom remaining in the selection of the basis is therefore given by

$$\begin{aligned} \bar{y}^a &= y^a + \mu x^a + \gamma z^a \\ \bar{x}^a &= x^a + \mu z^a \\ \bar{z}^a &= z^a \\ \bar{t}^a &= t^a. \end{aligned}$$

The scalars μ and γ will now be selected uniquely, so that the two remaining requirements of the quasi-orthogonal ennuple, namely $\bar{y}^a \bar{x}_a = 0$ and $\bar{y}^a \bar{y}_a = 0$, are satisfied:

$$\bar{y}^a \bar{x}_a = 2\mu + y^a x^a.$$

We shall select the scalar μ to be equal to $-\frac{1}{2}y^a x_a$:

$$\bar{y}^a \bar{y}_a = y^a y_a - 3\mu^2 + 2\gamma$$

and shall select the scalar γ such that this is identically zero. Hence a unique quasi-orthogonal ennuple exists, which gives a Jordan matrix representation for the linear operator h_{ab} on the space g_{ab}.

The scalars $\underset{\alpha\beta}{g}$ and $\underset{\alpha\beta}{h}$ are found:

$$\underset{\alpha\beta}{g} = g_{ab}\underset{(\alpha)}{v^a}\underset{(\beta)}{v^b} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\underset{\alpha\beta}{h} = h_{ab}\underset{(\alpha)}{v^a}\underset{(\beta)}{v^b} = \begin{pmatrix} 0 & A & 0 & 0 \\ A & 0 & 0 & 1 \\ 0 & 0 & A+B & 0 \\ 0 & 1 & 0 & A \end{pmatrix}.$$

These scalars, using the identities discussed in § 2, lead to

and

$$\left.\begin{aligned} g_{ab} &= x_a x_b + 2z_{(a}y_{b)} + t_a t_b \\ h_{ab} &= A g_{ab} + 2z_{(a}x_{b)} + B t_a t_b \end{aligned}\right\}. \tag{4.2}$$

The condition that the spaces h_{ab} and g_{ab} should have corresponding geodesics is that there exist a scalar μ such that

$$2\mu h_{ab1c} + 2h_{ab}\mu_{,c} + h_{bc}\mu_{,a} + h_{ca}\mu_{,b} = 0 \qquad \text{(Eisenhart 1926).}$$

Here, and throughout the remainder of this work, the covariant derivative is with respect to the metric g_{ab}. The conditions of integrability of these equations are $\bar{R}_{abcd} + \bar{R}_{bacd} = 0$ (Eisenhart 1926), where $\bar{R}_{abcd}$ is the Riemann tensor with respect to h_{ab}. These conditions are satisfied.

The components of this equation in the quasi-orthogonal basis are

$$2\mu \underset{\alpha\beta\rho}{h} + 2\underset{\alpha\beta}{h}\underset{\rho}{\mu} + \underset{\beta\rho}{h}\underset{\alpha}{\mu} + \underset{\rho\alpha}{h}\underset{\beta}{\mu} = 0 \tag{4.3}$$

where

$$\underset{\alpha\beta\rho}{h} = h_{ab1c}\underset{(\alpha)}{v^a}\underset{(\beta)}{v^b}\underset{(\rho)}{v^c}$$

and

$$\underset{\alpha}{\mu} = \mu_{,a}\underset{(\alpha)}{v^a}.$$

From the definitions

$$\underset{\alpha\beta}{h} = \underset{(\alpha\beta)}{h} \quad \text{and} \quad \underset{\alpha\beta\rho}{h} = \underset{(\alpha\beta)\rho}{h}$$

we have

$$\underset{\alpha\beta\rho}{h} = \begin{pmatrix} 0 & \underset{\rho}{A} + \underset{41\rho}{\gamma} & B\underset{31\rho}{\gamma} & 0 \\ \underset{\rho}{A} + \underset{41\rho}{\gamma} & 2\underset{42\rho}{\gamma} & B\underset{32\rho}{\gamma} + \underset{43\rho}{\gamma} & \underset{12\rho}{\gamma} \\ B\underset{31\rho}{\gamma} & B\underset{32\rho}{\gamma} + \underset{43\rho}{\gamma} & \underset{\rho}{A} + \underset{\rho}{B} & B\underset{34\rho}{\gamma} + \underset{13\rho}{\gamma} \\ 0 & \underset{12\rho}{\gamma} & B\underset{34\rho}{\gamma} + \underset{13\rho}{\gamma} & \underset{\rho}{A} + 2\underset{14\rho}{\gamma} \end{pmatrix}$$

where

$$\underset{\rho}{A} = A_{,a}\underset{(\rho)}{v^a} \quad \text{and} \quad \underset{\rho}{B} = B_{,a}\underset{(\rho)}{v^a}.$$

Equations (4.3) lead to certain conditions on the rotation coefficients and on the eigenvalues. These conditions are listed in appendix 1.

It is found that the congruence of z^a is null, geodetic, expansion free, hypersurface orthogonal and shear free. Hence, by the Goldberg–Sachs (1962) theorem, all vacuum metrics in this class are algebraically special. z^a need not be recurrent, so that the metrics need not be of Petrov type III or IId. The conditions on the rotation coefficients also give t^a to be hypersurface orthogonal. Using these and other properties of the rotation coefficients, the metrics of the spaces are now formulated.

Let z be a parameter along the congruence z^a, defined by $z_a = vz_{,a}$, where v is a scalar. Since there is no freedom in the choice of z^a, the scalar cannot be transformed away. To get rid of the scalar by introducing a degree of freedom in the choice of z^a, it is necessary to define the original basis with x^a being of convenient scalar magnitude, not necessarily unity. This brings in complications which are not worth the simplification obtained in the form of z_a. Since z^a is null, for a displacement along this congruence, $0 = g_{11}\, dz^2$, implying that g_{11} is zero. Let y be the parameter along the congruence of y^a; then g_{22} is zero also. The congruence t^a is hypersurface orthogonal, space-like. Let t be the parameter along this congruence and let x be a parameter along the curves of x^a. The vectors z_a, y_a, t_a and x_a in this coordinate system can be written

$$z_a = (V, 0, 0, 0), \qquad y_a = (y_1, y_2, y_3, y_4)$$
$$t_a = (0, 0, C, 0), \qquad x_a = (x_1, x_2, x_3, x_4)$$

where $y_1 \dots y_4$, $x_1 \dots x_4$ and C are as yet unknown scalars. The line element of the space g_{ab} in this coordinate system is

$$ds^2 = c^2\, dt^2 + D^2\, dx^2 + 2E\, dz\, dy + 2F\, dz\, dx + 2G\, dy\, dx$$

where D, E, F and G are scalars. The vector components and the metric coefficients may be related, using expression (4.2) for g_{ab}, to give

$$\left.\begin{array}{ll} G = 0 & \\ z_a = (V, 0, 0, 0), & y_a = \dfrac{1}{V}(-\tfrac{1}{2}Q^2, E, 0, F-DQ) \\ t_a = (0, 0, C, 0), & x_a = (Q, 0, 0, D) \end{array}\right\}. \tag{4.4}$$

Here Q is an unknown scalar. Both Q and V, if they exist, are unique.

The rotation coefficients may now be calculated. A knowledge of some of the rotation coefficients has already been used, of course, in the construction of the line element. Comparison will be made with the table of rotation coefficients to give the remaining information concerning the spaces. The table of rotation coefficients based on the above vectors is given in appendix 2.

An identity that can be used for simplifying some of the rotation coefficients is obtained from the knowledge that z^a is geodesic. $z_{a1b}z^b$ being proportional to z_a implies, in this coordinate system, that $\left\{ {c \atop 11} \right\} = 0$, $c \neq 1$, giving

$$FE_{,1} - EF_{,1} = 0 \quad \text{or} \quad F = 0.$$

It is found that the spaces can be represented by the following metrics:

space g_{ab}, $\quad ds^2 = C^2\, dt^2 + D^2\, dx^2 + 2E\, dz\, dy + 2F\, dz\, dx$

space h_{ab}, $\quad ds^2 = 2VQ\, dz^2 + C^2(A+B)\, dt^2 + AD^2\, dx^2 + 2AE\, dz\, dy + 2(AF + VD)\, dz\, dx.$

The conditions on the coefficients are as follows:

$$F = 0 \quad \text{or} \quad FE_{,1} - EF_{,1} = 0$$
$$V_{,2} = V_{,3} = C_{,2} = C_{,4} = D_{,2} = D_{,3} = E_{,3} = F_{,3} = Q_{,3} = 0$$
$$EQ_{,1} - QE_{,1} = \frac{3VQQ_{,2}}{2A}$$
$$\frac{A_{,1}}{2V} = \frac{V_{,4}}{VD} = \frac{2Q_{,2}}{E} = \frac{2(E_{,4} - F_{,2})}{ED} = -\frac{4ABC_{,1}}{3(A+B)CV}$$

$$B_{,1} = \frac{(3B-A)V_{,4}}{2AD}$$

$$\frac{Q_{,2}}{E}\{3VD-2A(QD-F)\} = 2A(Q_{,4}-D_{,1}).$$

$$\frac{Q_{,2}}{E}(9V-2AQ) = 2A\left(\frac{V_{,1}}{V}-\frac{QV_{,4}}{VD}-\frac{E_{,1}}{E}\right).$$

A and B are functions of z only. The following functions cannot take the value zero: A, B, $A+B$, E, D, V, C. z, y, t and x are labelled 1, 2, 3 and 4 coordinates, respectively. In order that the signature of both the spaces be $+2$, it is necessary and sufficient that all the functions appearing in the metric coefficients be real, that $A > 0$, $A+B > 0$.

5. [(3, 1)] Segrè characteristic

The Jordan canonical form is now

$$\begin{pmatrix} A & 1 & 0 & 0 \\ 0 & A & 1 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

A being the single repeated eigenvalue. The base vectors are defined by the chains

$$x^a = (h^a{}_b - A\delta^a_b)y^b$$

$$z^a = (h^a{}_b - A\delta^a_b)x^b$$

$$(h^a{}_b - A\delta^a_b)z^b = 0$$

$$(h^a{}_b - A\delta^a_b)t^b = 0.$$

The following identities are found in exactly the same manner as in § 4:

$$z^az_a = 0, \quad z^at_a = 0, \quad z^ay_a = 1, \quad z^ax_a = 0, \quad t^at_a = 1, \quad x^ax_a = 1, \quad x^at_a = 0.$$

To complete the quasi-orthogonal ennuple basis, the identities $y^ay_a = 0$, $y^at_a = 0$ and $y^ax_a = 0$ are still required. If we start with an arbitrary generalized eigenvector of rank 3, y^a, the most general transformation preserving this property is

$$\bar{y}^a = \rho y^a + \mu x^a + \gamma z^a + \eta t^a$$

where ρ, μ, γ and η are arbitrary scalers.

The remaining vectors in the chain, $\bar{x}^a$ and $\bar{z}^a$, are given by

$$\bar{x}^a = \rho x^a + \mu z^a$$

$$\bar{z}^a = \rho z^a.$$

The two-dimensional eigenspace allows $\bar{t}^a$, defined by $\bar{t}^a = \lambda t^a + \delta z^a$, to be an eigenvector. However, the restriction that $\bar{t}^a$ should have magnitude 1 causes $\lambda = 1$ and $\delta = 0$. The condition $\bar{y}^a\bar{y}_a = 1$ makes $\rho = 1$. All other identities are, of course, satisfied since $\bar{y}^a$ is an eigenvector of rank 3, $\bar{x}^a$ an eigenvector of rank 2 and $\bar{z}^a$ an eigenvector. Three degrees of freedom remain to construct an ennuple that satisfies the three remaining conditions:

$$\bar{y}^a\bar{x}_a = y^ax_a + 2\mu.$$

Let us select the appropriate scalar μ to make $\bar{y}^a\bar{x}_a = 0$.

$$\bar{y}^a\bar{t}_a = y^a t_a + \eta.$$

Here we select the appropriate η to make $\bar{y}^a\bar{t}_a = 0$:

$$\bar{y}^a\bar{y}_a = y^a y_a - 3\mu^2 - \eta^2 + \gamma.$$

Here again, by selecting the appropriate γ, $\bar{y}^a$ is null. Hence the quasi-orthogonal ennuple can be selected as the basis in this case. Here again it is unique. The representations of g_{ab} and h_{ab} are as in (4.2) with $B = 0$. The geodesic condition leads to the same identities as in the [3, 1] case. These now have to be simplified under the condition $B = 0$. $A = 0$, $B = 0$ need not be considered, as here the space h_{ab} becomes two-dimensional. $A \neq 0$, $B = 0$ in the identities give $\underset{1}{\mu} = \underset{2}{\mu} = \underset{3}{\mu} = \underset{4}{\mu} = 0$, implying that μ is constant. This in turn implies that A is constant.

The only possible non-identically zero rotation coefficients are $\underset{23\alpha}{\gamma}$ and $\underset{32\alpha}{\gamma}$, these being independent apart from the skew symmetry relationship. Hence the z^a congruence is again null, geodetic, expansion free, hypersurface orthogonal and shear free. By the Goldberg–Sachs theorem and the discussion in the introduction all vacuum metrics in this class are algebraically special.

The conditions on the rotation coefficients also imply that the congruence X^a is hypersurface orthogonal. The metrics are now constructed in a manner similar to those of the [3, 1] case. The analogy between the two cases is used in the construction. It is found that the spaces can be represented by the following metrics having signatures $+2$:

$$ds^2 = dx^2 + D^2\,dt^2 + 2E\,dz\,dy + 2F\,dz\,dt$$

and

$$ds^2 = A\,dx^2 + AD^2\,dt^2 + 2AE\,dz\,dy + 2AF\,dz\,dt + 2V\,dz\,dx.$$

The conditions on the coefficients are

$$D_{,2} = D_{,4} = E_{,4} = F_{,4} = E_{,3} - F_{,2} = 0$$

$$\frac{E_{,1}}{E} = \frac{V_{,1}}{V}.$$

D, V and E do not vanish, D, E, F and V are real valued, A is a real positive constant and V is a function of z only. Here z, y, t and x are labelled 1, 2, 3 and 4 coordinates, respectively. The two spaces then have corresponding geodesics.

6. Discussion

The Segrè class having simple elementary divisors and simple eigenvalues has been discussed by Eisenhart (1926). Levi-Civita (1896) has discussed the Segrè class having simple elementary divisors with repeated eigenvalues, when the fundamental forms are positive definite. An extension of this work to relativistic metrics and also the investigation of the [2, 1, 1] class and its sub-classes still need to be carried out. It is expected that spaces allowing corresponding geodesics exist in these categories. Since any physically realistic gravitational wave would have a certain amount of shear, it would be of interest to find metrics other than algebraically special ones having corresponding geodesics. These may exist in the classes still to be considered.

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Appendix 1

The conditions on the rotation coefficients are as follows:

$$\gamma_{\alpha\beta 1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{A}{2B}\gamma_{123} & -\dfrac{A}{3}\gamma_{122} \\ 0 & \dfrac{A}{2B}\gamma_{123} & 0 & 0 \\ 0 & \dfrac{A}{3}\gamma_{122} & 0 & 0 \end{pmatrix}$$

$$\gamma_{\alpha\beta 2} = \begin{pmatrix} 0 & \gamma_{122} & -\dfrac{A}{2B}\gamma_{123} & -\dfrac{A}{9}\gamma_{122} \\ -\gamma_{122} & 0 & -\dfrac{A+B}{2B^3}\gamma_{123} & 0 \\ \dfrac{A}{2B}\gamma_{123} & \dfrac{A+B}{2B^3}\gamma_{123} & 0 & \dfrac{A+B}{2B^2}\gamma_{123} \\ \dfrac{A}{9}\gamma_{122} & 0 & -\dfrac{A+B}{2B^2}\gamma_{123} & 0 \end{pmatrix}$$

$$\gamma_{\alpha\beta 3} = \begin{pmatrix} 0 & \gamma_{123} & 0 & 0 \\ -\gamma_{123} & 0 & -\dfrac{A+B}{3B}\gamma_{122} & 0 \\ 0 & \dfrac{A+B}{3B}\gamma_{122} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{\alpha\beta 4} = \begin{pmatrix} 0 & \dfrac{A}{3}\gamma_{122} & 0 & 0 \\ -\dfrac{A}{3}\gamma_{122} & 0 & -\dfrac{A+B}{2B^2} & -\tfrac{1}{3}\gamma_{122} \\ 0 & \dfrac{A+B}{2B^2}\gamma_{123} & 0 & \dfrac{A}{2B}\gamma_{123} \\ 0 & \tfrac{1}{3}\gamma_{122} & -\dfrac{A}{2B}\gamma_{123} & 0 \end{pmatrix}.$$

The conditions on the eigenvalues are as follows:

$$A_{,2} = A_{,4} = B_{,2} = B_{,4} = 0$$

$$A_{,1} - A_{,3}\frac{Q}{D} = \frac{8A}{9}\left(\frac{V_{,1}}{V} - \frac{E_{,1}}{E} - \frac{QV_{,4}}{2DV}\right)$$

$$A_{,3} = A\left(\frac{V_{,3}}{V} - \frac{E_{,3}}{2E}\right)$$

$$B_{,1}-\frac{B_{,3}Q}{D}=\frac{2}{9}(3B-A)\left(\frac{V_{,1}}{V}-\frac{E_{,1}}{E}-\frac{QV_{,4}}{2DV}\right)$$

$$B_{,3}=(A+2B)\left(\frac{V_{,3}}{V}-\frac{E_{,3}}{E}\right).$$

Appendix 2

The following rotation coefficients are identically zero: $\underset{\alpha\alpha\beta}{\gamma}$, for all α and β, by the skew symmetric property of rotation coefficients; $\underset{131}{\gamma}$, $\underset{141}{\gamma}$, $\underset{341}{\gamma}$, $\underset{133}{\gamma}$, $\underset{143}{\gamma}$, $\underset{134}{\gamma}$ and rotation coefficients obtained from these using the skew symmetric property.

The non-zero rotation coefficients are as follows:

$$\underset{121}{\gamma}=\frac{V_{,2}}{E}$$

$$\underset{231}{\gamma}=-\frac{E_{,3}}{2CE}$$

$$\underset{241}{\gamma}=-\frac{Q_{,2}}{E}+\frac{F_{,2}-E_{,4}}{2DE}$$

$$\underset{122}{\gamma}=\frac{V_{,1}}{V^2}-\frac{QV_{,4}}{V^2D}-\frac{E_{,1}}{EV}+\frac{(E_{,4}-F_{,2})Q}{EVD}$$

$$\underset{132}{\gamma}=-\frac{E_{,3}}{2EC}$$

$$\underset{142}{\gamma}=\frac{F_{,2}-E_{,4}}{2ED}$$

$$\underset{232}{\gamma}=-\frac{D_{,3}Q^2}{CDV^2}-\frac{E_{,3}Q(2F-QD)}{2CEDV^2}+\frac{F_{,3}Q}{CDV^2}$$

$$\underset{242}{\gamma}=\frac{QE_{,1}}{E}+\frac{Q^2(F_{,2}-E_{,4})}{2ED}-\frac{QD_{,1}}{D}-Q_{,1}-\frac{Q_{,2}(2QF-Q^2)}{2ED}+\frac{Q_{,4}Q}{D}$$

$$\underset{342}{\gamma}=\frac{E_{,3}(QD-F)}{2CDEV}+\frac{F_{,3}}{2CDV}-\frac{QD_{,3}}{CDV}$$

$$\underset{123}{\gamma}=\frac{V_{,3}}{VC}-\frac{E_{,3}}{2EC}$$

$$\underset{233}{\gamma}=\frac{C_{,1}}{CV}$$

$$\underset{343}{\gamma}=-\frac{C_{,4}}{CD}$$

$$\underset{124}{\gamma}=\frac{V_{,4}}{VD}-\frac{E_{,4}-F_{,2}}{2ED}$$

$$\underset{144}{\gamma}=\frac{VD_{,2}}{ED}$$

$$\underset{234}{\gamma} = \frac{E_{,3}(F-QD)}{2CDEV} - \frac{F_{,3}}{2DCV} + \frac{D_{,3}Q}{CDV}$$

$$\underset{244}{\gamma} = \frac{Q_{,2}(F-QD)}{DEV} - \frac{Q_{,4}}{VD} + \frac{D_{,1}}{VD}$$

$$\underset{344}{\gamma} = \frac{D_{,3}}{CD}.$$

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